Exercise 58.1

Let's begin by looking for possible interior solutions for Firm 1. For an interior solution the market price must be non-zero (otherwise Firm 1 could gain by deviating to $q_1 = 0$), and so we can write Firm 1's problem as choosing q_1 to maximize

$$\pi_1 = pq_1 - C_1(q_1) = (\alpha - q_1 - q_2)q_1 - c_1 q_1 = (\alpha - q_1 - q_2 - c_1)q_1.$$

Taking a partial derivative with respect to the choice variable and setting it equal to zero yields

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \alpha - 2q_1 - q_2 - c_1 = 0 \Longrightarrow q_1 = \frac{1}{2}(\alpha - q_2 - c_1).$$

So the only possible interior solution for Firm 1 that is a best response is

$$q_1 = \frac{1}{2}(\alpha - q_2 - c_1).$$

Of course, it is also possible that Firm 1's best response is to choose a corner solution ($q_1 = 0$ or $q_1 = \infty$). It is easy to see that $q_1 = \infty$ is never a best response, because in this case the market price will always be $p = 0$; Firm 1 will therefore be making negative profits and can gain by deviating to, e.g., $q_1 = 0$. So if Firm 1's action is a best response to Firm 2 then the only possibilities are

$$q_1 = 0 \text{ or } q_1 = \frac{1}{2}(\alpha - q_2 - c_1).$$

When will it choose one over the other? Well, the interior solution tells us to choose a negative value of q_1 (which is impossible) if $\alpha - q_2 - c_1 < 0$. And in fact we can see that $\alpha - q_2 - c_1 < 0$ implies that Firm 1 will make a negative profit if it chooses $q_1 > 0$, because then the market price $p = \alpha - q_1 - q_2$ will be less than c_1. So if $\alpha - q_2 - c_1 < 0$ then Firm 1's best response is $q_1 = 0$. Otherwise, Firm 1's profit from choosing $q_1 = \frac{1}{2}(\alpha - q_2 - c_1) \geq 0$ is

$$\pi_1 = (\alpha - q_1 - q_2 - c_1)q_1 = \left[\frac{1}{2}(\alpha - q_2 - c_1)\right]^2,$$

which is zero if $\alpha - q_2 - c_1 = 0$ and non-negative for $\alpha - q_2 - c_1 < 0$. So:

Firm 1's best response function is to choose

$$q_1 = 0 \textbf{ if } \alpha - q_2 - c_1 \leq 0 \quad \textbf{and} \quad q_1 = \frac{1}{2}(\alpha - q_2 - c_1) > 0 \textbf{ if } \alpha - q_2 - c_1 > 0.$$

A symmetric argument shows that

Firm 2's best response function is to choose

$$q_2 = 0 \textbf{ if } \alpha - q_1 - c_2 \leq 0 \quad \textbf{and} \quad q_2 = \frac{1}{2}(\alpha - q_1 - c_2) > 0 \textbf{ if } \alpha - q_1 - c_2 > 0.$$

We now have four possible Nash equilibriums to check: either both firms choose corner solutions ($q_1 = q_2 = 0$), or both firms choose interior solutions ($q_1 > 0, q_2 > 0$), or one firm chooses a corner solution and the other chooses an interior solution ($q_i = 0, q_j > 0$).

Option 1 Is there a Nash equilibrium with $q_1 = q_2 = 0$? No, because if $q_2 = 0$ then $\alpha - q_2 - c_1 = \alpha - c_1 > 0$, so Firm 1's best response is some $q_1 > 0$, e.g., the monopoly output $q_1 = \frac{1}{2}(\alpha - c_1)$. (Firm 2 can also gain by deviating here.)

Option 2 Is there a Nash equilibrium with $q_1 > 0$ and $q_2 > 0$? In this case we must have

$$q_1 = \frac{1}{2}(\alpha - q_2 - c_1) \text{ and } q_2 = \frac{1}{2}(\alpha - q_1 - c_2).$$

Solving these two equations simultaneously yields

$$q_1 = \frac{1}{3}(\alpha + c_2 - 2c_1) \text{ and } q_2 = \frac{1}{3}(\alpha + c_1 - 2c_2).$$

These are mutual best responses as long as

$$\alpha - q_2 - c_1 > 0 \text{ and } \alpha - q_1 - c_2 > 0,$$

which simplify to

$$\alpha - \frac{1}{3}(\alpha + c_1 - 2c_2) - c_1 > 0 \text{ and } \alpha - \frac{1}{3}(\alpha + c_2 - 2c_1) - c_2 > 0$$

and then to

$$\alpha - 2c_1 + c_2 > 0 \text{ and } \alpha - 2c_2 + c_1 > 0.$$

Because $c_1 > c_2$, both of these conditions will hold if and only if $\alpha - 2c_1 + c_2 > 0$, i.e., if and only if $c_1 < \frac{1}{2}(\alpha + c_2)$. In conclusion: if $c_1 < \frac{1}{2}(\alpha + c_2)$ then we have a Nash equilibrium at

$$q_1 = \frac{1}{2}(\alpha - q_2 - c_1) > 0 \text{ and } q_2 = \frac{1}{2}(\alpha - q_1 - c_2) > 0.$$

Option 3 Is there a Nash equilibrium in which Firm 1 chooses some $q_1 > 0$ and Firm 2 chooses $q_2 = 0$? Intuitively, this seems unlikely because Firm 2 is the low-cost producer. But let's check this formally with a proof by contradiction. Assume, then, that there is a Nash equilibrium in which Firm 2 chooses $q_2 = 0$ and Firm 1 chooses some $q_1 > 0$. We know that Firm 2 chooses $q_2 = 0$ only if $\alpha - q_1 - c_2 \leq 0$, i.e., only if $\alpha - q_1 \leq c_2$. But now we see that the market price $p = \alpha - q_1$ is less than c_2, and since $c_1 > c_2$ this means that the market price is less than c_1. This leads to a contradiction because Firm 1 is now making a negative profit and can gain by deviating alone to, e.g., $q_1 = 0$.

Option 4 Is there a Nash equilibrium in which Firm 2 chooses some $q_2 > 0$ and Firm 1 chooses $q_1 = 0$? Intuitively this seems possible because Firm 2, the low-cost producer, might be able to drive Firm 1 out of the market. What would such a Nash equilibrium look like? Well, Firm 2's best response function shows that it has to produce the monopoly level of output, $q_2 = \frac{1}{2}(\alpha - c_2)$. And Firm 1's choice of $q_1 = 0$ is a best response to this if and only if $\alpha - q_2 - c_1 \leq 0$. Substituting in for q_2 shows that this is a Nash equilibrium if and only if $\alpha - \frac{1}{2}(\alpha - c_2) - c_1 \leq 0$, i.e., if and only if $c_1 \geq \frac{1}{2}(c_2 + \alpha)$. In conclusion: we get a Nash equilibrium of $q_1 = 0$ and $q_2 = \frac{1}{2}(\alpha - c_2) > 0$ if and only if $c_1 \geq \frac{1}{2}(c_2 + \alpha)$.

To summarize: If $c_1 \geq \frac{1}{2}(c_2 + \alpha)$ then the unique Nash equilibrium is

$$q_1 = 0 \quad \text{and} \quad q_2 = \frac{1}{2}(\alpha - c_2) > 0.$$

If $c_1 < \frac{1}{2}(c_2 + \alpha)$ then the unique Nash equilibrium is

$$q_1 = \frac{1}{3}(\alpha + c_2 - 2c_1) > 0 \quad \text{and} \quad q_2 = \frac{1}{3}(\alpha + c_1 - 2c_2) > 0.$$

In either equilibrium, Firm 2 produces more. We can also see that reductions in c_2 increase Firm 2's output and reduce Firm 1's output (subject to the condition that $q_1 \geq 0$); because reductions in c_2 increase q_2 more than they reduce q_1, total output increases and consequently the market price falls.

Exercise 59.2

Let's begin by looking for possible interior solutions for Firm 1. For an interior solution the market price must be non-zero (otherwise Firm 1 could gain by deviating to $q_1 = 0$), and so we can write Firm 1's problem as choosing q_1 to maximize

$$\pi_1 = pq_1 - C_1(q_1) = (\alpha - q_1 - q_2)q_1 - cq_1 - F = (\alpha - q_1 - q_2 - c)q_1 - F.$$

Taking a partial derivative with respect to the choice variable and setting it equal to zero yields

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Longrightarrow \alpha - 2q_1 - q_2 - c = 0 \Longrightarrow q_1 = \frac{1}{2}(\alpha - q_2 - c).$$

So the only possible interior solution for Firm 1 that is a best response is

$$q_1 = \frac{1}{2}(\alpha - q_2 - c).$$

Of course, it is also possible that Firm 1's best response is to choose a corner solution ($q_1 = 0$ or $q_1 = \infty$). It is easy to see that $q_1 = \infty$ is never a best response, because in this case the market price will always be $p = 0$; Firm 1 will therefore be making negative profits and can gain by deviating to, e.g., $q_1 = 0$. So if Firm 1's action is a best response to Firm 2 then the only possibilities are

$$q_1 = 0 \quad \text{or} \quad q_1 = \frac{1}{2}(\alpha - q_2 - c).$$

When will it choose one over the other? Well, the interior solution tells us to choose a negative value of q_1 (which is impossible) if $\alpha - q_2 - c < 0$. But we also have to check what happens with profits. If Firm 1 chooses $q_1 = 0$ then it makes zero profit. If $\alpha - q_2 - c > 0$ then the interior solution tells us to choose a positive value of $q_1 = \frac{1}{2}(\alpha - q_2 - c)$, and this leads to profits of

$$\pi_1 = (\alpha - q_1 - q_2 - c)q_1 - F = \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2 - F.$$

So having $\alpha - q_2 - c > 0$ is not enough because if it's too small then the presence of the fixed cost F will result in Firm 1 making negative profits. The conclusion:

If $F \geq \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2$ **then Firm 1's best response is to choose** $q_1 = 0$**; if** $F \leq \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2$ **then Firm 1's best response is to choose** $q_1 = \frac{1}{2}(\alpha - q_2 - c)$. (Note that if $F = \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2$ then both $q_1 = 0$ and $q_1 = \frac{1}{2}(\alpha - q_2 - c)$ are best responses!)

A symmetric argument shows that

If $F \geq \left[\frac{1}{2}(\alpha - q_1 - c)\right]^2$ **then Firm 2's best response is to choose** $q_2 = 0$**; and if** $F \leq \left[\frac{1}{2}(\alpha - q_1 - c)\right]^2$ **then Firm 2's best response is to choose** $q_2 = \frac{1}{2}(\alpha - q_1 - c)$.

We now have four possible Nash equilibriums to check: either both firms choose corner solutions ($q_1 = q_2 = 0$), or both firms choose interior solutions ($q_1 > 0, q_2 > 0$), or one firm chooses a corner solution and the other chooses an interior solution ($q_i = 0, q_j > 0$).

Option 1 Is there a Nash equilibrium in which both firms choose corner solutions ($q_1 = q_2 = 0$)? Well, we see that these are mutual best responses if $F \geq \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2$ and $F \geq \left[\frac{1}{2}(\alpha - q_1 - c)\right]^2$. But since $q_1 = q_2 = 0$, these both simplify to the same condition: $F \geq \left[\frac{1}{2}(\alpha - c)\right]^2$. As long as this condition is met, we get a Nash equilibrium at $q_1 = q_2 = 0$. The intuition is clear: if F is "too big", then both firms will stay out of the market.

Option 2 Is there a Nash equilibrium in which both firms choose interior solutions ($q_1 > 0, q_2 > 0$)? In this case we must have

$$q_1 = \frac{1}{2}(\alpha - q_2 - c) \quad \text{and} \quad q_2 = \frac{1}{2}(\alpha - q_1 - c).$$

Solving these two equations simultaneously yields

$$q_1 = q_2 = \frac{1}{3}(\alpha - c).$$

These are mutual best responses if

$$F \leq \left[\frac{1}{2}(\alpha - q_1 - c)\right]^2 \quad \text{and} \quad F \leq \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2,$$

but since $q_1 = q_2$ these both simplify to the same condition:

$$F \leq \left[\frac{1}{2}\left(\alpha - \frac{1}{3}(\alpha - c) - c\right)\right]^2, \quad \text{i.e.,} \quad F \leq \left[\frac{1}{3}(\alpha - c)\right]^2.$$

As long as this condition is met, we get a Nash equilibrium at $q_1 = q_2 = \frac{1}{3}(\alpha - c)$. The intuition here also makes sense: if F is "small enough", then there's enough room for both firms in the market.

Option 3 Is there a Nash equilibrium in which Firm 1 chooses an interior solution $q_1 > 0$ and Firm 2 chooses $q_2 = 0$? Well, let's see. In order for Firm 1's choice of $q_1 > 0$ to be a best response we must have

$$F \leq \left[\frac{1}{2}(\alpha - q_2 - c)\right]^2 \quad \text{and} \quad q_1 = \frac{1}{2}(\alpha - q_2 - c).$$

Since $q_2 = 0$ these simplify to

$$F \leq \left[\frac{1}{2}(\alpha - c)\right]^2 \quad \text{and} \quad q_1 = \frac{1}{2}(\alpha - c).$$

And we know that Firm 2's choice of $q_2 = 0$ is a best response if $F \geq \left[\frac{1}{2}(\alpha - q_1 - c)\right]^2$. Substituting in for q_1 this yields

$$F \geq \left[\frac{1}{2}\left(\alpha - \frac{1}{2}(\alpha - c) - c\right)\right]^2, \quad \text{i.e.,} \quad F \geq \left[\frac{1}{2}\left(\alpha - \frac{1}{2}(\alpha - c) - c\right)\right]^2,$$

which simplifies to $F \geq \left[\frac{1}{4}(\alpha - c)\right]^2$. Combining all these conditions, we find that we have a Nash equilibrium at $q_1 = \frac{1}{2}(\alpha - c)$ and $q_2 = 0$ if $\left[\frac{1}{4}(\alpha - c)\right]^2 \leq F \leq \left[\frac{1}{2}(\alpha - c)\right]^2$.

Option 4 Is there a Nash equilibrium in which Firm 2 chooses an interior solution $q_2 > 0$ and Firm 1 chooses $q_1 = 0$? Yes, by symmetry we have a Nash equilibrium at $q_2 = \frac{1}{2}(\alpha - c)$ and $q_1 = 0$ if $\left[\frac{1}{4}(\alpha - c)\right]^2 \leq F \leq \left[\frac{1}{2}(\alpha - c)\right]^2$.

In conclusion:

- If F is "really big" (i.e., if $F \geq \left[\frac{1}{2}(\alpha - c)\right]^2$), then $q_1 = q_2 = 0$ is the only Nash equilibrium: F is so big that neither firm wants to be in the market.

- If F is "medium big" (i.e., if $\left[\frac{1}{3}(\alpha - c)\right]^2 \leq F \leq \left[\frac{1}{2}(\alpha - c)\right]^2$), then there are two Nash equilibriums, both of the form $q_i = \frac{1}{2}(\alpha - c)$, $q_j = 0$: in this case F is too big for both firms to be in the market but not big enough to keep both firms out.

- If F is "medium small" (i.e., if $\left[\frac{1}{4}(\alpha - c)\right]^2 \leq F \leq \left[\frac{1}{3}(\alpha - c)\right]^2$), then there are three Nash equilibriums: two of the form $q_i = \frac{1}{2}(\alpha - c)$, $q_j = 0$, and one with $q_1 = q_2 = \frac{1}{3}(\alpha - c)$: in this case F is small enough that both firms can be in the market, but not small enough to ensure that both firms *will* enter the market.

- If F is "really small" (i.e., if $F \leq \left[\frac{1}{4}(\alpha - c)\right]^2$), then $q_1 = q_2 = \frac{1}{3}(\alpha - c)$ is the unique Nash equilibrium: in this case F is so small that neither firm can keep the other out of the market.