Final Exam (75 Points Total) **Answer Key**

1. (a) Firm 1 chooses $q \geq 0$ to maximize profits, which are $\pi = q(100 - q)$. We either have a corner solution ($q = 0$ or $q = \infty$, the latter of which is clearly not profit-maximizing because profits are negative for $q > 100$) or an interior solution with $\frac{\partial \pi}{\partial q} = 0$, i.e., with $100 - 2q = 0$, i.e., with $q = 50$. If $q = 50$ then profits are $\pi = 50^2 = 2500$. This is larger than profits from choosing $q = 0$, so the optimal level of output for this firm is $q = 50$.

   (b) If $p_2 = 0$, Firm 2 is making negative profits and can gain by deviating alone, e.g., to $p_2 = 10$. If $0 < p_2 \leq 50$, Firm 1 can gain by deviating alone to $p_2 - \epsilon$ for sufficiently small ϵ. If $p_2 > 50$ and $p_1 \neq 50$, Firm 1 can gain by deviating alone to $p_1 = 50$. If $p_2 > 50$ and $p_1 = 50$, Firm 2 can gain by deviating alone, e.g., to 49.99.

   (c) If $p_2 > 50$ then Firm 1's best response is $p_1 = 50$. If $.01 < p_2 \leq 50$ then Firm 1's best response is $p_1 = p_2 - .01$. If $p_2 = .01$ then $p_1 = .01$ is Firm 1's best response. And if $p_2 = 0$ then any $p_1 \geq 0$ is a best response for Firm 1.

   (d) It might help to find Firm 1's best response function (which was determined in the previous problem) and Firm 2's best response, which is: if $p_1 > 45$, $p_2 = 45$ (its monopoly output) is the best response; if $10.01 < p_1 \leq 45$, $p_2 = p_1 - .01$ is the best response; if $p_1 = 10.01$, $p_2 = 10.01$ is the best response; if $p_1 = 10$, any $p_2 \geq p_1$ is a best response; if $p_1 < 10$, any $p_2 > p_1$ is a best response.

   So we put these together to find the Nash equilibrium(s). First, (p_1, p_2) is never a NE if $p_2 \leq p_1$; this is because Firm 1 can always gain by deviating to $p_1 = p_2 - .01$ unless $p_2 \leq .01$, but if $p_2 \leq .01$ then Firm 2 is making negative profit and can gain by deviating to, e.g., $p_2 > p_1$. So the only NEs feature $p_1 < p_2$. These must feature $p_1 \leq 10$ because otherwise Firm 2 could gain by deviating to $p_2 = p_1$. If $p_1 \leq 10$ then any NEs must also feature $p_1 = p_2 - .01$ because otherwise Firm 1 could gain by deviating to $p_1 = p_2 - .01$ or to $p_1 = 50$. So we get NEs at $(p_1, p_1 + .01)$ for $0 \leq p_1 \leq 10$.

   (e) Firm 1 chooses q_1 to maximize $\pi_1 = q_1(100 - q_1 - q_2)$. Firm 2 chooses q_2 to maximize $\pi_2 = q_2(100 - q_1 - q_2 - 10)$. Take derivatives with respect to choice variables and solve simultaneously to find an interior solution.

   (f) Firm 1 either has a corner solution ($q_1 = 0$ or $q_1 = \infty$) or an interior solution with $\frac{\partial \pi}{\partial q_1} = 0$, i.e., with $100 - 2q_1 - q_2 = 0$, i.e., with $q_1 = \frac{1}{2}(100 - q_2)$. So Firm 1's best response is $q_1 = \frac{1}{2}(100 - q_2)$ for $q_2 < 100$ and $q_1 = 0$ for $q_2 \geq 100$.

   Firm 2 either has a corner solution ($q_2 = 0$ or $q_2 = \infty$) or an interior solution with $\frac{\partial \pi}{\partial q_2} = 0$, i.e., with $100 - q_1 - 2q_2 - 10 = 0$, i.e., with

$q_2 = \frac{1}{2}(90 - q_1)$. So Firm 2's best response is $q_2 = \frac{1}{2}(90 - q_1)$ for $q_1 < 90$ and $q_2 = 0$ for $q_1 \geq 90$.

(g) Firm 1 chooses q_1 to maximize $\pi_1 = q_1(100 - q_1 - q_2)$ subject to Firm 2's best response function, $q_2 = \text{BRF}(q_1)$. Firm 2 chooses q_2 to maximize $\pi_2 = q_2(100 - q_1 - q_2 - 10)$. To find the interior solution, we solve Firm 2's problem to find Firm 2's best response function, $q_2 = \text{BRF}(q_1)$, and then plug this best response function into Firm 1's problem and differentiate to find Firm 1's optimal choice of q_1.

(h) Firm 2's best response function is the same as in the Cournot game above, $q_2 = \frac{1}{2}(90 - q_1)$. Firm 1 therefore chooses q_1 to maximize $\pi_1 = q_1(100 - q_1 - q_2)$ subject to Firm 2's best response function, i.e., chooses q_1 to maximize

$$\pi_1 = q_1(100 - q_1 - \frac{1}{2}(90 - q_1)) = q_1(55 - \frac{1}{2}q_1).$$

Differentiating with respect to q_1 we find an interior solution at $55 - q_1 = 0$, i.e., at $q_1 = 55$.

So the SPNE is $q_1 = 55$ and $q_2 = \frac{1}{2}(90 - q_1)$ for $q_1 < 90$ and $q_2 = 0$ for $q_1 \geq 90$. The actual output we will see is $q_1 = 55$ and $q_2 = \frac{35}{2}$.

(i) One such NE is for Firm 1 to produce $q_1 = 0$ and for Firm 2 to produce $q_2 = 45$ if $q_1 = 0$ and $q_2 = 100$. This NE is "wrong" because it involves Firm 2 making *non-credible threats*: if Firm 1 doesn't produce $q_1 = 0$, Firm 2 will not be optimizing by following this strategy.

2. There are plenty of examples here. Note that there is only one NE in such games: it's the pure strategy that leads to the Pareto inefficient outcome, hence the name "Prisoners' Dilemma".

3. (a) Player 1 chooses $p, 0 \leq p \leq 1$, to maximize

$$\pi_1 = pq(2) + p(1-q)(1) + (1-p)q(3) + (1-p)(1-q)(0) = p + 3q - 2pq.$$

At a maximum, either $p = 0$ or $p = 1$ or there is an interior solution, $0 < p < 1$, in which case

$$\frac{\partial \pi_1}{\partial p} = 0 \Longrightarrow 1 - 2q = 0 \Longrightarrow q = \frac{1}{2}.$$

Plugging $q = \frac{1}{2}$ into Player 1's objective function shows that $\pi_1 = \frac{3}{2}$ regardless of the choice of p, which means that any $p, 0 \leq p \leq 1$, is a best response if $q = \frac{1}{2}$. If $q \neq \frac{1}{2}$ then Player 1's best response is a corner solution, either $p = 0$ or $p = 1$. If $q = 0$ then $\pi_1 = p$, and if $q = 1$ then $\pi_1 = 3 - p$, so we can see that the rest of Player 1's best response function is to choose $p = 1$ if $q < \frac{1}{2}$ and to choose $p = 0$ if $q > \frac{1}{2}$.

(b) The pure strategy NEs are (U, D) and (D, U). (Note that neither of these are potential evolutionarily stable strategies because they are not symmetric.) The mixed strategy NE is given by $p = q = \frac{1}{2}$.

(c) Intuitively, an evolutionarily stable strategy means that mutants die. With mixed strategies, one interpretation is that all members of the population play U and D in the given proportions, and mutants who pick different proportions die. Another interpretation is that members of the population each play either U or D, but that the *fraction* of these two types is given by the mixed strategy; in this case, an ESS means that that population ratio is stable, i.e., that a slight increase in the proportion of U players or D players will not disrupt the equilibrium.

(d) We have to check two things to show that this is an ESS. First, is $p = q = \frac{1}{2}$ a Nash equilibrium? We know from above that the answer is yes, meaning that mutants cannot do better against dominant types than dominant types do against themselves. So far so good.

Second, we need to check that any mutant that does as well as dominant types against dominant types does less well than dominant types against fellow mutants. To check this, first note that any mutant (playing U with probability $x \neq \frac{1}{2}$, say) does just as well as dominant types against dominant types: this is because anything is a best response to $p = \frac{1}{2}$. Next, note that the payoff for dominant types against mutants is

$$\frac{1}{2}(x)(2) + \frac{1}{2}(1-x)(1) + \frac{1}{2}(x)(3) = 2x + \frac{1}{2}$$

and that the payoff for mutants against fellow mutants is

$$x^2(2) + x(1-x)(1) + (1-x)(x)(3) = 4x - 2x^2.$$

Mutants do as well or better than dominant types against fellow mutants if and only if $4x - 2x^2 \geq 2x + \frac{1}{2}$,, i.e., if and only if $2x^2 - 2x + \frac{1}{2} \leq 0$, i.e., if and only if $2\left(x - \frac{1}{2}\right)^2 \leq 0$. But this is true if and only if $x = \frac{1}{2}$; so any mutant with $x \neq \frac{1}{2}$ cannot do as well or better than dominant types against fellow mutants.

We conclude that $p = q = \frac{1}{2}$ is an ESS.