Optimization and Calculus

Derivatives in theory

The **derivative** of a function $f(x)$, written $\frac{d}{dx}\left[f(x)\right]$ or $\frac{d\,f(x)}{dx}$ or $f'(x)$, measures the *instantaneous rate of change* of $f(x)$:

$$\frac{d}{dx}\left[f(x)\right] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Intuitively, derivatives measures **slope**: $f'(x) = -3$ intuitively means that if x increases by 1 then $f(x)$ will decrease by 3. This intuition matches up with setting $h = 1$, which yields

$$f'(x) \approx \frac{f(x+1) - f(x)}{1} = f(x+1) - f(x).$$

All of the functions we use in this class have derivatives (i.e., are **differentiable**), which intuitively means that they are smooth and don't have kinks or discontinuities. The maximum and minimum values of such functions must either be **corner solutions**—such as $x = \infty$, $x = -\infty$, or (if we are trying to maximize $f(x)$ subject to $x \geq x_{min}$) $x = x_{min}$—or **interior solutions**. The vast majority of the problems in this class will have interior solutions.

At an interior maximum or minimum, the slope $f'(x)$ must be zero. Why? Well, if $f'(x) \neq 0$ then either $f'(x) > 0$ or $f'(x) < 0$. Intuitively, this means that you're on the side of a hill—if $f'(x) > 0$ you're going uphill, if $f'(x) < 0$ you're heading downhill—and if you're on the side of a hill then you're not at the top (a maximum) or at the bottom (a minimum). At the top and the bottom the slope $f'(x)$ will be zero.

To say the same thing in math: If $f'(x) \neq 0$ then either $f'(x) > 0$ or $f'(x) < 0$. If $f'(x) > 0$ then $f(x+h) > f(x)$ (this comes from the definition of derivative), so $f(x)$ isn't a maximum; and $f(x-h) < f(x)$ (this follows from continuity, i.e., the fact that $f(x)$ is a smooth function), so $f(x)$ isn't a minimum. Similarly, if $f'(x) < 0$ then $f(x+h) < f(x)$ (this comes from the definition of derivative), so $f(x)$ isn't a minimum; and $f(x-h) > f(x)$ (this follows from continuity), so $f(x)$ isn't a maximum. So the only possible (interior) maxima or minima must satisfy $f'(x) = 0$, which is called a **necessary first-order condition**.

In sum: to find candidate values for (interior) maxima or minima, simply take a derivative and set it equal to zero, i.e., find values of x that satisfy $f'(x) = 0$.

Such values do not *have* to be maxima or minima: the condition $f'(x) = 0$ is *necessary* but not *sufficient*. This is a more advanced topic that we will not get into in this course, but for an example consider $f(x) = x^3$. Setting the derivative $(3x^2)$ equal to zero has only one solution: $x = 0$. But $x = 0$ is neither a minimum nor a maximum value of $f(x) = x^3$. The **sufficient second-order condition** has to do with the second derivative (i.e., the derivative of the derivative, written $f''(x)$). For a maximum, the sufficient second-order condition is $f''(x) < 0$; this guarantees that we're on a hill, so together with $f'(x) = 0$ it guarantees that we're on the top of the hill. For a minimum, the sufficient second-order condition is $f''(x) > 0$; this guarantees that we're in a valley, so together with $f'(x) = 0$ it guarantees that we're at the bottom of the valley.)

Partial derivatives

For functions of two or more variables such as $f(x, y)$, it is often useful to see what happens when we change one variable (say, x) without changing the other variables. (For example, what happens if we walk in the north-south direction without changing our east-west position?) What we end up with is the *partial derivative with respect to* x of the function $f(x, y)$, written $\frac{\partial}{\partial x}[f(x, y)]$ or $f_x(x, y)$:

$$\frac{\partial}{\partial x}[f(x, y)] = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Partial derivatives measure rates of change or slopes *in a given direction*: $f_x(x, y) = 3y$ intuitively means that if x increases by 1 and y doesn't change then $f(x, y)$ will increase by $3y$. Note that "regular" derivatives and partial derivatives mean the same thing for a function of only one variable: $\frac{d}{dx}[f(x)] = \frac{\partial}{\partial x}[f(x)]$.

At an (interior) maximum or minimum of a smooth function, the slope must be zero *in all directions*. In other words, the necessary first-order conditions are that *all* partials must be zero: $f_x(x, y) = 0$, $f_y(x, y) = 0$, etc. Why? For the same reasons we gave before: if one of the partials—say, $f_y(x, y)$—is not zero, then moving in the y direction takes us up or down the side of a hill, and so we cannot be at a maximum or minimum value of the function $f(x, y)$.

In sum: to find candidate values for (interior) maxima or minima, simply take partial derivatives with respect to all the variables and set them equal to zero, e.g., find values of x and y that simultaneously satisfy $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

As before, these conditions are necessary but not sufficient. This is an even more advanced topic than before, and we will not get into it in this course; all I will tell you here is that (1) the sufficient conditions for a maximum include $f_{xx} < 0$ and $f_{yy} < 0$, *but these aren't enough*, (2) you can find the sufficient conditions in most advanced textbooks, e.g., Silberberg and Suen's *The Structure of Economics*, and (3) an interesting example to consider is $f(x, y) = \frac{\cos(x)}{\cos(y)}$ around the point $(0, 0)$.

One final point: Single variable derivatives can be thought of as a degenerate case of partial derivatives: there is no reason we can't write $f_x(x)$ instead of $f'(x)$ or $\frac{\partial}{\partial x} f(x)$ instead of $\frac{d}{dx} f(x)$. All of these terms measure the same thing: the rate of change of the function $f(x)$ in the x direction.

Derivatives in practice

To see how to calculate derivatives, let's start out with a very simple function: the constant function $f(x) = c$, e.g., $f(x) = 2$. We can calculate the derivative of this function from the definition:

$$\frac{d}{dx}(c) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

So the derivative of $f(x) = c$ is $\frac{d}{dx}(c) = 0$. Note that *all* values of x are candidate values for maxima and/or minima. Can you see why?[1]

Another simple function is $f(x) = x$. Again, we can calculate the derivative from the definition:

$$\frac{d}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

So the derivative of $f(x) = x$ is $\frac{d}{dx}(x) = 1$. Note that *no* values of the function $f(x) = x$ are candidate values for maxima or minima. Can you see why?[2]

A final simple derivative involves the function $g(x) = c \cdot f(x)$ where c is a constant and $f(x)$ is any function:

$$\frac{d}{dx}[c \cdot f(x)] = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The last term on the right hand side is simply the derivative of $f(x)$, so the derivative of $g(x) = c \cdot f(x)$ is $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$.

More complicated derivatives

To differentiate (i.e., calculate the derivative of) a more complicated function, use various differentiation rules to methodically break down your problem until you get an expression involving the derivatives of the simple functions shown above.

The most common rules are those involving the three main binary operations: addition, multiplication, and exponentiation.

[1] Answer: All values of x *are* maxima; all values are minima, too! Any x you pick gives you $f(x) = c$, which is both the best and the worst you can get.

[2] Answer: There are no interior maxima or minima of the function $f(x) = x$.

- **Addition** $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}[f(x)] + \dfrac{d}{dx}[g(x)].$

  Example: $\dfrac{d}{dx}[x+2] = \dfrac{d}{dx}[x] + \dfrac{d}{dx}[2] = 1 + 0 = 1.$

  Example: $\dfrac{d}{dx}\left[3x^2(x+2)+2x\right] = \dfrac{d}{dx}\left[3x^2(x+2)\right] + \dfrac{d}{dx}[2x]\,.$

- **Multiplication** $\dfrac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\dfrac{d}{dx}[g(x)] + g(x)\cdot\dfrac{d}{dx}[f(x)]\,.$

  Example: $\dfrac{d}{dx}[3x] = 3\cdot\dfrac{d}{dx}[x] + x\cdot\dfrac{d}{dx}[3] = 3(1) + x(0) = 3.$

  (Note: this also follows from the result that $\dfrac{d}{dx}[c\cdot f(x)] = c\cdot\dfrac{d}{dx}[f(x)]$.)

  Example: $\dfrac{d}{dx}[x(x+2)] = x\cdot\dfrac{d}{dx}[(x+2)] + (x+2)\cdot\dfrac{d}{dx}[x] = 2x+2.$

  Example: $\dfrac{d}{dx}\left[3x^2(x+2)\right] = 3x^2\cdot\dfrac{d}{dx}[(x+2)] + (x+2)\cdot\dfrac{d}{dx}\left[3x^2\right]\,.$

- **Exponentiation** $\dfrac{d}{dx}[f(x)^a] = a\cdot f(x)^{a-1}\cdot\dfrac{d}{dx}[f(x)]\,.$

  Example: $\dfrac{d}{dx}\left[(x+2)^2\right] = 2(x+2)^1\cdot\dfrac{d}{dx}[x+2] = 2(x+2)(1) = 2(x+2).$

  Example: $\dfrac{d}{dx}\left[(2x+2)^{\frac{1}{2}}\right] = \dfrac{1}{2}(2x+2)^{-\frac{1}{2}}\cdot\dfrac{d}{dx}[2x+2] = (2x+2)^{-\frac{1}{2}}.$

Putting all these together, we can calculate lots of messy derivatives:

$$\begin{aligned}
\frac{d}{dx}\left[3x^2(x+2)+2x\right] &= \frac{d}{dx}\left[3x^2(x+2)\right] + \frac{d}{dx}[2x] \\
&= 3x^2\cdot\frac{d}{dx}[x+2] + (x+2)\cdot\frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}[2x] \\
&= 3x^2(1) + (x+2)(6x) + 2 \\
&= 9x^2 + 12x + 2
\end{aligned}$$

Subtraction and division

The rule for addition also works for subtraction, and can be seen by treating $f(x) - g(x)$ as $f(x) + (-1)\cdot g(x)$ and using the rules for addition and multiplication. Less obviously, the rule for multiplication takes care of division: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{d}{dx}\left[f(x)\cdot g(x)^{-1}\right]$. Applying the product and exponentiation rules to this yields the quotient rule,[3]

[3] Popularly remembered as $d\left[\dfrac{\text{Hi}}{\text{Ho}}\right] = \dfrac{\text{Ho}\cdot d\text{Hi} - \text{Hi}\cdot d\text{Ho}}{\text{Ho}\cdot\text{Ho}}.$

- **Division** $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{[g(x)]^2}.$

  Example: $\dfrac{d}{dx}\left[\dfrac{3x^2+2}{-e^x}\right] = \dfrac{-e^2 \cdot \frac{d}{dx}[3x^2+2] - (3x^2+2) \cdot \frac{d}{dx}[-e^x]}{[-e^x]^2}.$

Exponents

If you're confused about what's going on with the quotient rule, you may find value in the following rules about exponents, which we will use frequently:

$$x^a \cdot x^b = x^{a+b} \qquad (x^a)^b = x^{ab} \qquad x^{-a} = \frac{1}{x^a} \qquad (xy)^a = x^a y^a$$

Examples: $2^2 \cdot 2^3 = 2^5$, $(2^2)^3 = 2^6$, $2^{-2} = \frac{1}{4}$, $(2 \cdot 3)^2 = 2^2 \cdot 3^2$.

Other differentiation rules: e^x and $\ln(x)$

You won't need the chain rule, but you may need the rules for derivatives involving the exponential function e^x and the natural logarithm function $\ln(x)$. (Recall that e and ln are inverses of each other, so that $e^{(\ln x)} = \ln(e^x) = x$.)

- **The exponential function** $\dfrac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \cdot \dfrac{d}{dx}\left[f(x)\right].$

  Example: $\dfrac{d}{dx}\left[e^x\right] = e^x \cdot \dfrac{d}{dx}\left[x\right] = e^x.$

  Example: $\dfrac{d}{dx}\left[e^{3x^2+2}\right] = e^{3x^2+2} \cdot \dfrac{d}{dx}\left[3x^2+2\right] = e^{3x^2+2} \cdot (6x).$

- **The natural logarithm function** $\dfrac{d}{dx}\left[\ln f(x)\right] = \dfrac{1}{f(x)} \cdot \dfrac{d}{dx}\left[f(x)\right].$

  Example: $\dfrac{d}{dx}\left[\ln x\right] = \dfrac{1}{x} \cdot \dfrac{d}{dx}\left[x\right] = \dfrac{1}{x}.$

  Example: $\dfrac{d}{dx}\left[\ln(3x^2+2)\right] = \dfrac{1}{3x^2+2} \cdot \dfrac{d}{dx}\left[3x^2+2\right] = \dfrac{1}{3x^2+2}(6x).$

Partial derivatives

Calculating partial derivatives (say, with respect to x) is easy: just treat all the other variables as constants while applying all of the rules from above. So, for example,

$$\begin{aligned}
\frac{\partial}{\partial x}\left[3x^2 y + 2e^{xy} - 2y\right] &= \frac{\partial}{\partial x}\left[3x^2 y\right] + \frac{\partial}{\partial x}\left[2e^{xy}\right] - \frac{\partial}{\partial x}\left[2y\right] \\
&= 3y\frac{\partial}{\partial x}\left[x^2\right] + 2e^{xy}\frac{\partial}{\partial x}\left[xy\right] - 0 \\
&= 6xy + 2ye^{xy}.
\end{aligned}$$

Note that the partial derivative $f_x(x,y)$ is a function of both x *and* y. This simply says that the rate of change with respect to x of the function $f(x,y)$ depends on where you are in both the x direction and the y direction.

We can also take a partial derivative with respect to y of the same function:

$$\begin{aligned}
\frac{\partial}{\partial y}\left[3x^2y + 2e^{xy} - 2y\right] &= \frac{\partial}{\partial y}\left[3x^2y\right] + \frac{\partial}{\partial y}\left[2e^{xy}\right] - \frac{\partial}{\partial y}\left[2y\right] \\
&= 3x^2\frac{\partial}{\partial y}\left[y\right] + 2e^{xy}\frac{\partial}{\partial y}\left[xy\right] - 2 \\
&= 3x^2 + 2xe^{xy} - 2.
\end{aligned}$$

Again, this partial derivative is a function of both x and y.

Integration

The integral of a function, written $\int_a^b f(x)\,dx$, measures the *area* under the function $f(x)$ between the points a and b. We won't use integrals much, but they are related to derivatives by the Fundamental Theorem(s) of Calculus:

$$\int_a^b \frac{d}{dx}\left[f(x)\right]\,dx = f(b) - f(a) \qquad \frac{d}{ds}\left[\int_a^s f(x)\,dx\right] = f(s)$$

Example: $\displaystyle\int_0^1 x\,dx = \int_0^1 \frac{d}{dx}\left[\frac{1}{2}x^2\right]\,dx = \frac{1}{2}(1^2) - \frac{1}{2}(0^2) = \frac{1}{2}$

Example: $\displaystyle\frac{d}{ds}\left[\int_0^s x\,dx\right] = s.$